

# On the Spectrum of a Model Operator in Fock Space

Tulkin H. Rasulov, Mukhiddin I. Muminov and Mahir Hasanov

## Abstract

A model operator  $H$  associated to a system describing four particles in interaction, without conservation of the number of particles, is considered. We describe the essential spectrum of  $H$  by the spectrum of the channel operators and prove the Hunziker-van Winter-Zhislin (HWZ) theorem for the operator  $H$ . We also give some variational principles for boundaries of the essential spectrum and interior eigenvalues.

AMS subject Classifications: Primary 81Q10; Secondary 35P20, 47N50.

**Key words and phrases:** Fock space, model operator, conservation of number of particles, channel operators, Hilbert-Schmidt class, Faddeev-Yakubovskii type system of integral equations, essential spectrum, variational principles.

## 1 INTRODUCTION

Spectral properties of multi-particle continuous Schrödinger operators are sufficiently well studied in [8]. As is well-known, the theorem on the location of the essential spectrum of multi-particle Hamiltonians was named the HWZ theorem in [5, 20] to the honor of Hunziker [11], van Winter [24] and Zhislin [25]. A lattice analogue of this theorem for the four-particle Schrödinger operator is proved in [1, 16].

The effective description of the location of the essential spectrum of electromagnetic Schrödinger operators on  $\mathbf{R}^N$  is obtained in [17]. The well-known methods for the investigation of the location of essential spectra of Schrödinger operators are Weyl criterion for the one particle problem and the HWZ theorem for multiparticle problems, the modern proof of which is based on the Ruelle-Simon partition of unity. In [18] by means of the limit operators method the essential spectrum of discrete Schrödinger operators on lattice  $\mathbf{Z}^N$  is studied. This method has been applied by one of the authors to describe the essential spectrum of continuous electromagnetic Schrödinger operators, square-root Klein-Gordon operators and Dirac operators under quite weak assumptions on the behavior of the magnetic and electric potential at infinity.

The systems considered above have a fixed number of quasi-particles. In statistical physics [14], solid-state physics [15] and the theory of quantum fields [9], one considers systems, where the number of quasi-particles is bounded, but not fixed. The study of these systems usually is reduced to the study of the spectral properties of self-adjoint operators, acting in the *cut subspace*  $\mathcal{H}^{(N)}$  of Fock space, consisting of  $n \leq N$  particles [9, 14, 15, 21, 26].

In [21] geometric and commutator techniques have been developed in order to find the location of the spectrum and to prove absence of singular continuous spectrum for

Hamiltonians without conservation of the particle number. The model operators acting in  $\mathcal{H}^{(3)}$  were well studied in [2, 3, 12, 13, 19, 22].

In the present paper we consider a model operator  $H$  associated to a system describing four particles in interaction, without conservation of the number of particles, acting in  $\mathcal{H}^{(4)}$ . For the study of location of the essential spectrum of  $H$  we introduce the channel operators and prove that the essential spectrum of  $H$  is the union of spectra of channel operators. The channel operators have a more simple structure than  $H$ . The two-particle, three-particle and four-particle branches of the essential spectrum of  $H$  are singled out. We also prove the HWZ theorem on the location of the essential spectrum of  $H$ . A variational approach to find boundaries of essential spectrum and some interior eigenvalues is given at the end of the paper.

The plan of the present paper is as follows.

Section 1 is an introduction to the whole work. In Section 2 the model operator  $H$  is described as a bounded self-adjoint operator in  $\mathcal{H}^{(4)}$  and the main results of the present paper are formulated. In Section 3 we study spectrum of channel operators by the spectrum of corresponding families of operators. In Section 4 we obtain analogue of the Faddeev-Yakubovskii type system of integral equations for the eigenvectors of  $H$ . Section 5 is devoted to the proof of the main results of the present paper (Theorems 2.1 and 2.2). In Section 6 we apply some results from classical variational theory and the variational theory of the spectrum of operator pencils to the model operator  $H$ .

Throughout the present paper we adopt the following convention: Denote by  $\mathbf{T}^\nu$  the  $\nu$ -dimensional torus, the cube  $(-\pi, \pi]^\nu$  with appropriately identified sides. The torus  $\mathbf{T}^\nu$  will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on the  $\nu$ -dimensional space  $\mathbf{R}^\nu$  modulo  $(2\pi\mathbf{Z})^\nu$ .

## 2 THE MODEL OPERATOR AND STATEMENTS OF THE MAIN RESULTS

Let us introduce some notations used in this work. Let  $\mathbf{C}$  be the field of complex numbers,  $(\mathbf{T}^\nu)^n$ ,  $n = 1, 2, 3$  be the Cartesian  $n$ th power of  $\mathbf{T}^\nu$  and  $L_2((\mathbf{T}^\nu)^n)$ ,  $n = 1, 2, 3$  be the Hilbert space of square-integrable (complex) functions defined on  $(\mathbf{T}^\nu)^n$ ,  $n = 1, 2, 3$ .

Denote

$$\mathcal{H}_0 = \mathbf{C}, \mathcal{H}_1 = L_2(\mathbf{T}^\nu), \mathcal{H}_2 = L_2((\mathbf{T}^\nu)^2), \mathcal{H}_3 = L_2((\mathbf{T}^\nu)^3),$$

$$\mathcal{H}^{(n,m)} = \bigoplus_{i=n}^m \mathcal{H}_i, \quad 0 \leq n < m \leq 3.$$

The Hilbert space  $\mathcal{H}^{(4)} \equiv \mathcal{H}^{(0,3)}$  is called "four-particle cut subspace" of Fock space.

Let the model operator  $H$  act in the Hilbert space  $\mathcal{H}^{(0,3)}$  as a matrix operator

$$H = \begin{pmatrix} H_{00} & H_{01} & 0 & 0 \\ H_{10} & H_{11} & H_{12} & 0 \\ 0 & H_{21} & H_{22} & H_{23} \\ 0 & 0 & H_{32} & H_{33} \end{pmatrix}, \quad (2.1)$$

where its components  $H_{ij} : \mathcal{H}_j \rightarrow \mathcal{H}_i$ ,  $i, j = 0, 1, 2, 3$  are defined by the rule

$$(H_{00}f_0)_0 = w_0 f_0, \quad (H_{01}f_1)_0 = \int_{\mathbf{T}^\nu} v_1(s) f_1(s) ds, \quad (H_{10}f_0)_1(p) = v_1(p) f_0,$$

$$(H_{11}f_1)_1(p) = w_1(p)f_1(p), \quad (H_{12}f_2)_1(p) = \int_{\mathbf{T}^\nu} v_2(s)f_2(p, s)ds,$$

$$(H_{21}f_1)_2(p, q) = v_2(q)f_1(p), \quad H_{22} = H_{22}^0 - V_{21} - V_{22},$$

$$(H_{22}^0f_2)_2(p, q) = w_2(p, q)f_2(p, q), \quad (V_{21}f_2)_2(p, q) = v_{21}(p) \int_{\mathbf{T}^\nu} v_{21}(s)f_2(s, q)ds,$$

$$(V_{22}f_2)_2(p, q) = v_{22}(q) \int_{\mathbf{T}^\nu} v_{22}(s)f_2(p, s)ds, \quad (H_{23}f_3)_2(p, q) = \int_{\mathbf{T}^\nu} v_3(s)f_3(p, q, s)ds,$$

$$(H_{32}f_2)_3(p, q, t) = v_3(t)f_2(p, q), \quad (H_{33}f_3)_3(p, q, t) = w_3(p, q, t)f_3(p, q, t).$$

Here  $f_i \in \mathcal{H}_i$ ,  $i = 0, 1, 2, 3$ ,  $w_0$  is a real number,  $v_i(\cdot)$ ,  $i = 1, 2, 3$ ,  $v_{2j}(\cdot)$ ,  $j = 1, 2$ ,  $w_1(\cdot)$  are real-valued continuous functions on  $\mathbf{T}^\nu$  and  $w_2(\cdot, \cdot)$  resp.  $w_3(\cdot, \cdot, \cdot)$  is a real-valued continuous function on  $(\mathbf{T}^\nu)^2$  resp.  $(\mathbf{T}^\nu)^3$ .

Under these assumptions the operator  $H$  is bounded and self-adjoint in  $\mathcal{H}^{(0,3)}$ .

We remark that the operators  $H_{01}$ ,  $H_{12}$  and  $H_{23}$  resp.  $H_{10}$ ,  $H_{21}$  and  $H_{32}$  defined in the Fock space are called annihilation resp. creation operators.

To formulate the main results of the present paper we introduce the following channel operators  $H_n$ ,  $n = 1, 3$  resp.  $H_2$  acting in  $\mathcal{H}^{(2,3)}$  resp.  $\mathcal{H}^{(1,3)}$  by the following formula

$$H_1 = \begin{pmatrix} H_{22}^0 - V_{21} & H_{23} \\ H_{32} & H_{33} \end{pmatrix}, \quad H_3 = \begin{pmatrix} H_{22}^0 & H_{23} \\ H_{32} & H_{33} \end{pmatrix}$$

resp.

$$H_2 = \begin{pmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22}^0 - V_{22} & H_{23} \\ 0 & H_{32} & H_{33} \end{pmatrix}.$$

Now we give the main results of the paper (for the proof see Section 5).

The essential spectrum of the operator  $H$  can be precisely described as well as in the following

**Theorem 2.1** *The essential spectrum  $\sigma_{ess}(H)$  of the operator  $H$  is the union of spectra of channel operators  $H_1$ ,  $H_2$  and  $H_3$ , i. e., the equality*

$$\sigma_{ess}(H) = \bigcup_{n=1}^3 \sigma(H_n)$$

*holds, where  $\sigma(H_n)$ ,  $n = 1, 2, 3$  stands the spectrum of the operator  $H_n$ ,  $n = 1, 2, 3$ .*

The following theorem shows that the least element of the essential spectrum of  $H$  belongs to the spectrum of channel operator  $H_1$  or  $H_2$ .

**Theorem 2.2** *(a HWZ theorem). The following equality*

$$\min \sigma_{ess}(H) = \min\{\min \sigma(H_1), \min \sigma(H_2)\}$$

*holds.*

### 3 THE SPECTRUM OF THE CHANNEL OPERATORS

In this section we describe the spectrum of the channel operators  $H_n$ ,  $n = 1, 2$  resp.  $H_3$  by the spectrum of the family of operators  $h_n(p)$ ,  $p \in \mathbf{T}^\nu$ ,  $n = 1, 2$  resp.  $h_3(p, q)$ ,  $p, q \in \mathbf{T}^\nu$  defined below.

First we consider the operator  $H_3$ , which commutes with any multiplication operator  $U_\alpha^{(3)}$  by the bounded function  $\alpha(\cdot, \cdot)$  on  $(\mathbf{T}^\nu)^2$

$$U_\alpha^{(3)} \begin{pmatrix} g_2(p, q) \\ g_3(p, q, t) \end{pmatrix} = \begin{pmatrix} \alpha(p, q)g_2(p, q) \\ \alpha(p, q)g_3(p, q, t) \end{pmatrix}, \quad \begin{pmatrix} g_2 \\ g_3 \end{pmatrix} \in \mathcal{H}^{(2,3)}.$$

Therefore the decomposition of the space  $\mathcal{H}^{(2,3)}$  into the direct integral

$$\mathcal{H}^{(0,1)} = \int_{(\mathbf{T}^\nu)^2} \oplus \mathcal{H}^{(2,3)} dp dq$$

yields the decomposition into the direct integral

$$H_3 = \int_{(\mathbf{T}^\nu)^2} \oplus h_3(p, q) dp dq, \quad (3.1)$$

where a family of the generalized Friedrichs models  $h_3(p, q)$ ,  $p, q \in \mathbf{T}^\nu$  acts in  $\mathcal{H}^{(0,1)}$  as

$$h_3(p, q) = \begin{pmatrix} h_{00}^{(3)}(p, q) & h_{01}^{(3)} \\ h_{10}^{(3)} & h_{11}^{(3)}(p, q) \end{pmatrix}.$$

Here

$$(h_{00}^{(3)}(p, q)f_0)_0 = w_2(p, q)f_0, \quad (h_{01}^{(3)}f_1)_0 = \int_{\mathbf{T}^\nu} v_3(s)f_1(s)ds,$$

$$(h_{10}^{(3)}f_0)_1(t) = v_3(t)f_0, \quad (h_{11}^{(3)}(p, q)f_1)_1(t) = w_3(p, q, t)f_1(t).$$

In analogy with the operator  $H_3$  one can give the decomposition

$$H_n = \int_{\mathbf{T}^\nu} \oplus h_n(p) dp, \quad n = 1, 2, \quad (3.2)$$

where a family of the operators  $h_1(p)$ ,  $p \in \mathbf{T}^\nu$  resp.  $h_2(p)$ ,  $p \in \mathbf{T}^\nu$  acts in  $\mathcal{H}^{(1,2)}$  resp.  $\mathcal{H}^{(0,2)}$  as

$$h_1(p) = \begin{pmatrix} h_{11}^{(1)}(p) & h_{12}^{(1)} \\ h_{21}^{(1)} & h_{22}^{(1)}(p) \end{pmatrix} \quad \text{resp.} \quad h_2(p) = \begin{pmatrix} h_{00}^{(2)}(p) & h_{01}^{(2)} & 0 \\ h_{10}^{(2)} & h_{11}^{(2)}(p) & h_{12}^{(1)} \\ 0 & h_{21}^{(1)} & h_{22}^{(1)}(p) \end{pmatrix}$$

with the entries

$$(h_{11}^{(1)}(p)f_1)_1(q) = w_2(p, q)f_1(q) - v_{21}(q) \int_{\mathbf{T}^\nu} v_{21}(s)f_1(s)ds, \quad (h_{12}^{(1)}f_2)_1(q) = \int_{\mathbf{T}^\nu} v_3(s)f_2(q, s)ds,$$

$$(h_{21}^{(1)}f_1)_2(q, t) = v_3(t)f_1(q), \quad (h_{22}^{(1)}(p)f_2)_2(q, t) = w_3(p, q, t)f_2(q, t),$$

$$(h_{00}^{(2)}(p)f_0)_0 = w_1(p)f_0, (h_{01}^{(2)}f_1)_0 = \int_{\mathbf{T}^\nu} v_2(s)f_1(s)ds, (h_{10}^{(2)}f_0)_1(q) = v_2(q)f_0,$$

$$(h_{11}^{(2)}(p)f_1)_1(q) = w_2(p,q)f_1(q) - v_{22}(q) \int_{\mathbf{T}^\nu} v_{22}(s)f_1(s)ds.$$

Let us introduce the notations

$$m = \min_{p,q,t \in \mathbf{T}^\nu} w_3(p,q,t), \quad M = \max_{p,q,t \in \mathbf{T}^\nu} w_3(p,q,t),$$

$$\sigma_{four}(H_n) = [m; M], \quad n = 1, 2, 3,$$

$$\sigma_{three}(H_n) = \bigcup_{p,q \in \mathbf{T}^\nu} \sigma_{disc}(h_3(p,q)), \quad n = 1, 2, 3,$$

$$\sigma_{two}(H_n) = \bigcup_{p \in \mathbf{T}^\nu} \sigma_{disc}(h_n(p)), \quad n = 1, 2.$$

The spectrum of the operators  $H_n$ ,  $n = 1, 2, 3$  can be precisely described as well as in the following

**Theorem 3.1** *The following equalities hold:*

- (i)  $\sigma(H_1) = \sigma_{two}(H_1) \cup \sigma_{three}(H_1) \cup \sigma_{four}(H_1);$
- (ii)  $\sigma(H_2) = \sigma_{two}(H_2) \cup \sigma_{three}(H_2) \cup \sigma_{four}(H_2);$
- (iii)  $\sigma(H_3) = \sigma_{three}(H_3) \cup \sigma_{four}(H_3).$

Before proving the Theorem 3.1 we introduce a new subsets of the essential spectrum of  $H$ .

**Definition 3.2** *The sets  $\sigma_{two}(H) = \sigma_{two}(H_1) \cup \sigma_{two}(H_2)$ ,  $\sigma_{three}(H) = \sigma_{three}(H_3)$  and  $\sigma_{four}(H) = \sigma_{four}(H_3)$  are called two-particle, three-particle and four-particle branches of the essential spectrum of  $H$ , respectively.*

We start the proof of the Theorem 3.1 with the following auxiliary statements.

Let the operator  $h_3^0(p,q)$ ,  $p, q \in \mathbf{T}^\nu$  acts in  $\mathcal{H}^{(0,1)}$  as

$$h_3^0(p,q) = \begin{pmatrix} 0 & 0 \\ 0 & h_{11}^{(3)}(p,q) \end{pmatrix}, \quad p, q \in \mathbf{T}^\nu.$$

The perturbation  $h_3(p,q) - h_3^0(p,q)$ ,  $p, q \in \mathbf{T}^\nu$  of the operator  $h_3^0(p,q)$ ,  $p, q \in \mathbf{T}^\nu$  is a self-adjoint operator of rank 2. Therefore in accordance with the invariance of the essential spectrum under finite rank perturbations the essential spectrum  $\sigma_{ess}(h_3(p,q))$  of  $h_3(p,q)$ ,  $p, q \in \mathbf{T}^\nu$  fills the following interval on the real axis:

$$\sigma_{ess}(h_3(p,q)) = [m_3(p,q); M_3(p,q)],$$

where the numbers  $m_3(p,q)$  and  $M_3(p,q)$  are defined by

$$m_3(p,q) = \min_{t \in \mathbf{T}^\nu} w_3(p,q,t), \quad M_3(p,q) = \max_{t \in \mathbf{T}^\nu} w_3(p,q,t).$$

**Remark 3.3** We remark that for some  $p, q \in \mathbf{T}^\nu$  the essential spectrum of  $h_3(p, q)$  may degenerate to the set consisting of the unique point  $\{m_3(p, q)\}$  and hence we cannot state that the essential spectrum of  $h_3(p, q)$  is absolutely continuous for any  $p, q \in \mathbf{T}^\nu$ . For example, this is the case if the function  $w_3(\cdot, \cdot, \cdot)$  is of the form

$$w_3(p, q, t) = \varepsilon(p) + \varepsilon(q + t) + \varepsilon(t),$$

where  $p = q = \underbrace{(\pi, \dots, \pi)}_\nu \in \mathbf{T}^\nu$  and

$$\varepsilon(t) = \nu - \sum_{i=1}^{\nu} \cos t_i, \quad t = (t_1, t_2, \dots, t_\nu) \in \mathbf{T}^\nu.$$

For any fixing  $p, q \in \mathbf{T}^\nu$  we define an analytic function  $\Delta_3(p, q; \cdot)$  (the Fredholm determinant associated with the operator  $h_3(p, q)$ ,  $p, q \in \mathbf{T}^\nu$ ) in  $\mathbf{C} \setminus \sigma_{ess}(h_3(p, q))$  by

$$\Delta_3(p, q; z) = w_2(p, q) - z - \int_{\mathbf{T}^\nu} \frac{v_3^2(s) ds}{w_3(p, q, s) - z}.$$

The following lemma established a connection between of eigenvalues of  $h_3(p, q)$ ,  $p, q \in \mathbf{T}^\nu$  and the zeroes of the function  $\Delta_3(p, q; \cdot)$ ,  $p, q \in \mathbf{T}^\nu$ .

**Lemma 3.4** For any fixing  $p, q \in \mathbf{T}^\nu$  the number  $z \in \mathbf{C} \setminus \sigma_{ess}(h_3(p, q))$  is an eigenvalue of the operator  $h_3(p, q)$ ,  $p, q \in \mathbf{T}^\nu$  if and only if  $\Delta_3(p, q; z) = 0$ .

**Proof.** "Only If Part." Let for any fixing  $p, q \in \mathbf{T}^\nu$  the number  $z \in \mathbf{C} \setminus \sigma_{ess}(h_3(p, q))$  be an eigenvalue of the operator  $h_3(p, q)$ ,  $p, q \in \mathbf{T}^\nu$  and  $f = (f_0, f_1) \in \mathcal{H}^{(0,1)}$  be the corresponding eigenvector, i. e., the equation  $h_3(p, q)f = zf$  or the system of equations

$$\begin{cases} (w_2(p, q) - z)f_0 + \int_{\mathbf{T}^\nu} v_3(s)f_1(s)ds = 0 \\ v_3(t)f_0 + (w_3(p, q, t) - z)f_1(t) = 0 \end{cases} \quad (3.3)$$

has a nontrivial solution  $f = (f_0, f_1) \in \mathcal{H}^{(0,1)}$ .

Since  $z \in \mathbf{C} \setminus \sigma_{ess}(h_3(p, q))$  from the second equation of the system (3.3) we find

$$f_1(t) = -\frac{v_3(t)f_0}{w_3(p, q, t) - z}. \quad (3.4)$$

Substituting the expression (3.4) for  $f_1$  into the first equation of the system (3.3), we get  $f_0\Delta_3(p, q; z) = 0$ . If  $f_0 = 0$ , then  $f_1(q) = 0$ . This contradicts the fact that  $f = (f_0, f_1)$  is an eigenvector the operator  $h_3(p, q)$ . Thus,  $\Delta_3(p, q; z) = 0$ .

"If Part." Let for some  $z \in \mathbf{C} \setminus \sigma_{ess}(h_3(p, q))$  the equality  $\Delta_3(p, q; z) = 0$  hold. It is easy to show that the vector-function  $f = (f_0, f_1) \in \mathcal{H}^{(0,1)}$  is an eigenvector of the operator  $h_3(p, q)$ ,  $p, q \in \mathbf{T}^\nu$  corresponding to the eigenvalue  $z \in \mathbf{C} \setminus \sigma_{ess}(h_3(p, q))$ , where  $f_0 = \text{const} \neq 0$  and  $f_1$  is defined by (3.4).  $\square$

From the Lemma 3.4 immediately follows the following equality

$$\sigma_{disc}(h_3(p, q)) = \{z \in \mathbf{C} \setminus \sigma_{ess}(h_3(p, q)) : \Delta_3(p, q; z) = 0\}, \quad p, q \in \mathbf{T}^\nu. \quad (3.5)$$

Using definitions of the operators  $h_1(p)$ ,  $p \in \mathbf{T}^\nu$  and  $h_2(p)$ ,  $p \in \mathbf{T}^\nu$  we obtain that for any  $p \in \mathbf{T}^\nu$  the equality  $\sigma_{ess}(h_1(p)) = \sigma_{ess}(h_2(p))$  holds.

For any fixing  $p \in \mathbf{T}^\nu$  we define an analytic function  $\Delta_1(p; \cdot)$  resp.  $\Delta_2(p; z)$  (the Fredholm determinant associated with the operator  $h_1(p)$ ,  $p \in \mathbf{T}^\nu$  resp.  $h_2(p)$ ,  $p \in \mathbf{T}^\nu$ ) in  $\mathbf{C} \setminus \sigma_{ess}(h_1(p))$  by

$$\Delta_1(p; z) = 1 - \int_{\mathbf{T}^\nu} \frac{v_{21}^2(s) ds}{\Delta_3(p, s; z)}$$

resp.

$$\Delta_2(p; z) = \left( 1 - \int_{\mathbf{T}^\nu} \frac{v_{22}^2(s) ds}{\Delta_3(p, s; z)} \right) \left( w_1(p) - z - \int_{\mathbf{T}^\nu} \frac{v_2^2(s) ds}{\Delta_3(p, s; z)} \right) - \left( \int_{\mathbf{T}^\nu} \frac{v_2(s) v_{22}(s) ds}{\Delta_3(p, s; z)} \right)^2.$$

Analogously to (3.5) one can derive the equalities

$$\sigma_{disc}(h_1(p)) = \{z \in \mathbf{C} \setminus \sigma_{ess}(h_1(p)) : \Delta_1(p; z) = 0\}, \quad p \in \mathbf{T}^\nu \quad (3.6)$$

and

$$\sigma_{disc}(h_2(p)) = \{z \in \mathbf{C} \setminus \sigma_{ess}(h_2(p)) : \Delta_2(p; z) = 0\}, \quad p \in \mathbf{T}^\nu. \quad (3.7)$$

**Proof of Theorem 3.1.** The assertions of the Theorem 3.1 follows from the representations (3.1), (3.2) and the theorem on decomposable operators (see [20]) and the equalities (3.5)-(3.7).  $\square$

**Corollary 3.5** *The following inclusion*

$$\sigma(H_3) \subset \sigma(H_1) \cup \sigma(H_2)$$

*holds.*

The proof of the Corollary 3.5 immediately follows from the Theorem 3.1.

## 4 THE FADDEEV-YAKUBOVSKII TYPE SYSTEM OF INTEGRAL EQUATIONS AND THE OPERATOR $T(z)$

In this section we derive an analog of the Faddeev-Yakubovskii type system of integral equations for the eigenvectors, corresponding to the eigenvalues lying outside of the essential spectrum of the operator  $H$ .

Let us introduce the notations

$$\overline{\mathcal{H}}_0 = \mathcal{H}_0, \quad \overline{\mathcal{H}}_1 = \overline{\mathcal{H}}_2 = \overline{\mathcal{H}}_3 = \mathcal{H}_1 \quad \text{and} \quad \overline{\mathcal{H}} = \bigoplus_{i=0}^3 \overline{\mathcal{H}}_i.$$

For each  $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  let the operator matrices  $A(z)$  and  $K(z)$  act in the Hilbert space  $\overline{\mathcal{H}}$  as

$$A(z) = \begin{pmatrix} A_{00}(z) & 0 & 0 & 0 \\ 0 & A_{11}(z) & 0 & A_{13}(z) \\ 0 & 0 & A_{22}(z) & 0 \\ 0 & A_{31}(z) & 0 & A_{33}(z) \end{pmatrix},$$

$$K(z) = \begin{pmatrix} K_{00}(z) & K_{01}(z) & 0 & 0 \\ K_{10}(z) & 0 & K_{12}(z) & 0 \\ 0 & K_{21}(z) & 0 & K_{23}(z) \\ 0 & 0 & K_{32}(z) & 0 \end{pmatrix},$$

where  $A_{ij}(z) : \overline{\mathcal{H}}_j \rightarrow \overline{\mathcal{H}}_i$ ,  $i, j = 0, 1, 2, 3$  is the multiplication operator by the function  $a_{ij}(p; z)$ :

$$a_{00}(p; z) \equiv 1, \quad a_{11}(p; z) = w_1(p) - z - \int_{\mathbf{T}^\nu} \frac{v_2^2(s)ds}{\Delta_3(p, s; z)},$$

$$a_{13}(p; z) \equiv a_{31}(p; z) = \int_{\mathbf{T}^\nu} \frac{v_2(s)v_{22}(s)ds}{\Delta_3(p, s; z)},$$

$$a_{22}(p; z) = 1 - \int_{\mathbf{T}^\nu} \frac{v_{21}^2(s)ds}{\Delta_3(s, p; z)}, \quad a_{33}(p; z) = 1 - \int_{\mathbf{T}^\nu} \frac{v_{22}^2(s)ds}{\Delta_3(p, s; z)},$$

and the operators  $K_{ij}(z) : \overline{\mathcal{H}}_j \rightarrow \overline{\mathcal{H}}_i$ ,  $i, j = 0, 1, 2, 3$  are defined as

$$(K_{00}(z)\psi_0)_0 = (w_0 - z + 1)\psi_0, \quad K_{01}(z) \equiv H_{01}, \quad K_{10}(z) \equiv -H_{10},$$

$$(K_{12}(z)\psi_2)_1(p) = -v_{21}(p) \int_{\mathbf{T}^\nu} \frac{v_2(s)\psi_2(s)ds}{\Delta_3(p, s; z)},$$

$$(K_{21}(z)\psi_1)_2(p) = -v_2(p) \int_{\mathbf{T}^\nu} \frac{v_{21}(s)\psi_1(s)ds}{\Delta_3(s, p; z)},$$

$$(K_{23}(z)\psi_3)_2(p) = v_{22}(p) \int_{\mathbf{T}^\nu} \frac{v_{21}(s)\psi_3(s)ds}{\Delta_3(s, p; z)},$$

$$(K_{32}(z)\psi_2)_3(p) = v_{21}(p) \int_{\mathbf{T}^\nu} \frac{v_{22}(s)\psi_2(s)ds}{\Delta_3(p, s; z)}.$$

We note that for each  $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  the operators  $K_{ij}(z)$ ,  $i, j = 0, 1, 2$  belong to the Hilbert-Schmidt class and therefore  $K(z)$  is a compact operator.

**Lemma 4.1** *For each  $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  the operator  $A(z)$  is bounded and invertible and the inverse operator  $A^{-1}(z)$  is given by*

$$A^{-1}(z) = \begin{pmatrix} B_{00}(z) & 0 & 0 & 0 \\ 0 & B_{11}(z) & 0 & B_{13}(z) \\ 0 & 0 & B_{22}(z) & 0 \\ 0 & B_{31}(z) & 0 & B_{33}(z) \end{pmatrix},$$

where  $B_{ij}(z) : \overline{\mathcal{H}}_j \rightarrow \overline{\mathcal{H}}_i$ ,  $i, j = 0, 1, 2, 3$  is the multiplication operator by the function  $b_{ij}(p; z)$ :

$$b_{00}(p; z) \equiv 1, \quad b_{11}(p; z) = \frac{a_{33}(p; z)}{\Delta_2(p; z)}, \quad b_{13}(p; z) = b_{31}(p; z) = -\frac{a_{13}(p; z)}{\Delta_2(p; z)},$$

$$b_{22}(p; z) = \frac{1}{\Delta_1(p; z)}, \quad b_{33}(p; z) = \frac{a_{11}(p; z)}{\Delta_2(p; z)}.$$



**Proof.** By the definition  $A(z)$  is the multiplication operator by the matrix  $A(p; z)$ , where

$$A(p; z) = \begin{pmatrix} a_{00}(p; z) & 0 & 0 & 0 \\ 0 & a_{11}(p; z) & 0 & a_{13}(p; z) \\ 0 & 0 & a_{22}(p; z) & 0 \\ 0 & a_{31}(p; z) & 0 & a_{33}(p; z) \end{pmatrix}.$$

Obviously, for each  $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  the matrix-valued function  $A(\cdot; z)$  is a continuous on  $\mathbf{T}^\nu$ . This implies that  $A(z)$  is bounded. Since  $\det A(p; z) = \Delta_1(p; z)\Delta_2(p; z)$  and  $z \notin \sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3)$ , we have that  $\det A(p; z) \neq 0$ . Therefore for each  $p \in \mathbf{T}^\nu$  and  $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  the matrix  $A(p; z)$  is invertible and its inverse matrix has the form

$$A^{-1}(p; z) = \begin{pmatrix} a_{00}(p; z) & 0 & 0 & 0 \\ 0 & \frac{a_{33}(p; z)}{\Delta_2(p; z)} & 0 & -\frac{a_{13}(p; z)}{\Delta_2(p; z)} \\ 0 & 0 & \frac{1}{\Delta_1(p; z)} & 0 \\ 0 & -\frac{a_{13}(p; z)}{\Delta_2(p; z)} & 0 & -\frac{a_{13}(p; z)}{\Delta_2(p; z)} \end{pmatrix}.$$

Then for each  $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  the matrix-valued function  $A^{-1}(\cdot; z)$  is a continuous on  $\mathbf{T}^\nu$ . Let  $A^{-1}(z)$  be the multiplication operator by the matrix  $A^{-1}(p; z)$  acting in  $\overline{\mathcal{H}}$ . It is easy to show that  $A^{-1}(z)$  is the inverse of  $A(z)$ .  $\square$

Since for each  $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  the operator  $A(z)$  is invertible, for such  $z$  we can define the operator  $T(z) = A^{-1}(z)K(z)$ .

The following lemma established a connection between of eigenvalues of  $H$  and  $T(z)$ .

**Lemma 4.2** *The number  $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  is an eigenvalue of the operator  $H$  if and only if the number  $\lambda = 1$  is an eigenvalue the operator  $T(z)$ .*

**Proof.** Let  $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  be an eigenvalue of the operator  $H$  and  $f = (f_0, f_1, f_2, f_3) \in \mathcal{H}^{(0,3)}$  be the corresponding eigenvector, that is, the equation  $Hf = zf$  or the system of equations

$$\begin{aligned} ((H_{00} - zI_0)f_0)_0 + (H_{01}f_1)_0 &= 0; \\ (H_{10}f_0)_1(p) + ((H_{11} - zI_1)f_1)_1(p) + (H_{12}f_2)_1(p) &= 0; \\ (H_{21}f_1)_2(p, q) + ((H_{22} - zI_2)f_2)_2(p, q) + (H_{23}f_3)_2(p, q) &= 0; \\ (H_{32}f_2)_3(p, q, t) + ((H_{33} - zI_3)f_3)_3(p, q, t) &= 0 \end{aligned} \tag{4.1}$$

has a nontrivial solution  $f = (f_0, f_1, f_2, f_3) \in \mathcal{H}^{(0,3)}$ , where  $I_i, i = \overline{0, 3}$  is an identity operator in  $\mathcal{H}_i, i = \overline{0, 3}$ . Since  $z \notin \sigma_{four}(H_3)$ , from the fourth equation of the system (4.1) for  $f_3$  we have

$$f_3(p, q, t) = -\frac{v_3(t)f_2(p, q)}{w_3(p, q, t) - z}. \tag{4.2}$$

Substituting the expression (4.2) for  $f_3$  into the third equation of the system (4.1) we obtain that the system of equations

$$\begin{aligned} ((H_{00} - zI_0)f_0)_0 + (H_{01}f_1)_0 &= 0; \\ (H_{10}f_0)_1(p) + ((H_{11} - zI_1)f_1)_1(p) + (H_{12}f_2)_1(p) &= 0; \end{aligned} \tag{4.3}$$

$$(H_{21}f_1)_2(p, q) + ((H_{22} - zI_2 - H_{23}R_{33}(z)H_{32})f_2)_2(p, q) = 0$$

has a nontrivial solution if and only if the system of equations (4.1) has a nontrivial solution, where  $R_{33}(z)$  is the resolvent of  $H_{33}$ .

Since  $z \notin \sigma_{three}(H_3)$  from the third equation of system (4.3) for  $f_2$  we have

$$f_2(p, q) = -\frac{v_2(q)f_1(p)}{\Delta_3(p, q; z)} + \frac{v_{21}(p)c_1(q) + v_{22}(q)c_2(p)}{\Delta_3(p, q; z)}, \quad (4.4)$$

where

$$c_1(q) = \int_{\mathbf{T}^\nu} v_{21}(s)f_2(s, q)ds, \quad (4.5)$$

$$c_2(p) = \int_{\mathbf{T}^\nu} v_{22}(s)f_2(p, s)ds. \quad (4.6)$$

Next we transform the system (4.3) using  $f_0, f_1, c_1, c_2$ . Substituting the expression (4.4) for  $f_2$  into the second equation of the system (4.3) and the equalities (4.5), (4.6) we obtain that the system of equations

$$\begin{aligned} f_0 &= (w_0 - z + 1)f_0 + \int_{\mathbf{T}^\nu} v_1(s)f_1(s)ds; \\ \left( w_1(p) - z - \int_{\mathbf{T}^\nu} \frac{v_2^2(s)ds}{\Delta_3(p, s; z)} \right) f_1(p) &+ \int_{\mathbf{T}^\nu} \frac{v_2(s)v_{22}(s)ds}{\Delta_3(p, s; z)} c_2(p) = \\ &- v_1(p)f_0 - v_{21}(p) \int_{\mathbf{T}^\nu} \frac{v_2(s)c_1(s)ds}{\Delta_3(p, s; z)}; \\ \left( 1 - \int_{\mathbf{T}^\nu} \frac{v_{21}^2(s)ds}{\Delta_3(s, q; z)} \right) c_1(q) &= -v_2(q) \int_{\mathbf{T}^\nu} \frac{v_{21}(s)f_1(s)ds}{\Delta_3(s, q; z)} + v_{22}(q) \int_{\mathbf{T}^\nu} \frac{v_{21}(s)c_2(s)ds}{\Delta_3(s, q; z)}; \\ \int_{\mathbf{T}^\nu} \frac{v_2(s)v_{22}(s)ds}{\Delta_3(p, s; z)} f_1(p) &+ \left( 1 - \int_{\mathbf{T}^\nu} \frac{v_{22}^2(s)ds}{\Delta_3(p, s; z)} \right) c_2(p) = v_{21}(p) \int_{\mathbf{T}^\nu} \frac{v_{22}(s)c_1(s)ds}{\Delta_3(p, s; z)} \end{aligned} \quad (4.7)$$

or the equation

$$A(z)\psi = K(z)\psi, \quad \psi = (f_0, f_1, c_1, c_2) \in \overline{\mathcal{H}}$$

has a nontrivial solution if and only if the system of equations (4.3) has a nontrivial solution.

By the Lemma 4.1 for each  $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  the operator  $A(z)$  is invertible and hence the equation

$$\psi = A^{-1}(z)K(z)\psi$$

or

$$\psi = T(z)\psi$$

has a nontrivial solution if and only if the system of equations (4.7) has a nontrivial solution.  $\square$

**Remark 4.3** We point out that the equation  $T(z)g = g$  is an analogues of the Faddeev-Yakubovskii type system of integral equations for eigenvectors of the operator  $H$ .

## 5 THE PROOF OF THE MAIN RESULTS

In this section applying the Weyl criterion and the Faddeev-Yakubovskii type system of integral equations we prove Theorem 2.1, then the proof of Theorem 2.2 will be follow from Theorems 2.1 and 3.1.

**Proof of Theorem 2.1.** The inclusion  $\sigma(H_3) \subset \sigma_{ess}(H)$  can be proven quite similarly to the corresponding inclusion of [12]. We prove that  $\sigma(H_1) \cup \sigma(H_2) \subset \sigma_{ess}(H)$ .

The set  $\sigma(H_1) \cup \sigma(H_2)$  we rewrite in the form

$$\sigma(H_1) \cup \sigma(H_2) = \sigma_{two}(H_1) \cup \sigma_{two}(H_2) \cup \sigma(H_3).$$

Let  $z_0$  be an arbitrary point of  $\sigma(H_1) \cup \sigma(H_2)$ . There are two cases possible:

$$1) z_0 \in \sigma(H_3),$$

$$2) z_0 \notin \sigma(H_3).$$

If  $z_0 \in \sigma(H_3)$ , then  $z_0 \in \sigma_{ess}(H)$ . Let  $z_0 \in (\sigma_{two}(H_1) \cup \sigma_{two}(H_2)) \setminus \sigma(H_3)$ . By the definition of  $\sigma_{two}(H_1) \cup \sigma_{two}(H_2)$ , there exists a point  $p_0 \in \mathbf{T}^\nu$  such that  $\Delta_1(p_0; z_0)\Delta_2(p_0; z_0) = 0$ . Then the system of homogenous linear equations

$$\begin{aligned} l_0 &= 0; \\ \left( w_1(p_0) - z_0 - \int_{\mathbf{T}^\nu} \frac{v_2^2(s)ds}{\Delta_3(p_0, s; z_0)} \right) l_1 + \int_{\mathbf{T}^\nu} \frac{v_2(s)v_{22}(s)ds}{\Delta_3(p_0, s; z_0)} l_3 &= 0; \\ \left( 1 - \int_{\mathbf{T}^\nu} \frac{v_{21}^2(s)ds}{\Delta_3(p_0, s; z_0)} \right) l_2 &= 0; \\ \int_{\mathbf{T}^\nu} \frac{v_2(s)v_{22}(s)ds}{\Delta_3(p_0, s; z_0)} l_1 + \left( 1 - \int_{\mathbf{T}^\nu} \frac{v_{22}^2(s)ds}{\Delta_3(p_0, s; z_0)} \right) l_3 &= 0 \end{aligned} \tag{5.1}$$

has an infinite number of solutions on  $\mathbf{C}^4$ , where  $\mathbf{C}^4$  is the Cartesian fourth power of  $\mathbf{C}$ . It is easy to verify that there exists a nontrivial solution  $\mathbf{l} = (0, l_1, l_2, l_3) \in \mathbf{C}^4$  of system of equations (5.1) satisfying one of the following conditions:

1. If  $\Delta_2(p_0; z_0) = 0$ , then either  $l_1 \neq 0$  and  $l_2 = 0$  or  $l_1 = 0$ ,  $l_2 = 0$  and  $l_3 \neq 0$ .
2. If  $\Delta_1(p_0; z_0) = 0$ , then  $l_2 \neq 0$  and  $l_1 = l_3 = 0$ .

The system of equations (5.1) can be written in the form

$$A(p_0; z_0)l = 0, \quad l = (0, l_1, l_2, l_3) \in \mathbf{C}^4.$$

Let  $\chi_{V_n}(\cdot)$  be the characteristic function of the set

$$V_n(p_0) = \left\{ p \in \mathbf{T}^\nu : \frac{1}{n+1} < |p - p_0| < \frac{1}{n} \right\}, \quad n = 1, 2, \dots$$

and  $\mu(V_n(p_0))$  be the Lebesgue measure of the set  $V_n(p_0)$ .

We choose a sequence of orthogonal vector-functions  $\{f^{(n)}\}$  as

$$f^{(n)} = \begin{pmatrix} 0 \\ f_1^{(n)}(p) \\ f_2^{(n)}(p, q) \\ f_3^{(n)}(p, q, t) \end{pmatrix},$$

where

$$\begin{aligned} f_1^{(n)}(p) &= \psi_1^{(n)}(p), \quad p \in \mathbf{T}^\nu, \\ f_2^{(n)}(p, q) &= -\frac{v_2(q)\psi_1^{(n)}(p)}{\Delta_3(p, q; z_0)} + \frac{v_{21}(p)\psi_2^{(n)}(q) + v_{22}(q)\psi_3^{(n)}(p)}{\Delta_3(p, q; z_0)}, \quad p, q \in \mathbf{T}^\nu, \\ f_3^{(n)}(p, q, t) &= -\frac{v_3(t)f_2^{(n)}(p, q)}{w_3(p, q, t) - z_0}, \quad p, q, t \in \mathbf{T}^\nu, \\ \psi_i^{(n)}(p) &= l_i k_n(p) \chi_{V_n}(p) (\mu(V_n(p_0)))^{-1/2}, \quad i = 1, 2, 3. \end{aligned}$$

Here  $\{k_n\} \subset L_2(\mathbf{T}^\nu)$  is to found from the orthogonality condition for  $\{f^{(n)}\}$ , i. e.,

$$\begin{aligned} (f^{(n)}, f^{(m)}) &= \frac{l_2}{\sqrt{\mu(V_n(p_0))}\sqrt{\mu(V_m(p_0))}} \int_{V_n(p_0)} \int_{V_m(p_0)} \left( 1 + \int_{\mathbf{T}^\nu} \frac{v_3^2(t)dt}{(w_3(p, q, t) - z_0)^2} \right) \times \\ &\left[ \frac{v_{21}(p)(l_3 v_{22}(q) - l_1 v_2(q))}{\Delta_3^2(p, q; z_0)} + \frac{v_{21}(q)(l_3 v_{22}(p) - l_1 v_2(p))}{\Delta_3^2(p, q; z_0)} \right] k_n(p) k_m(q) dp dq = 0, \quad n \neq m. \end{aligned}$$

The existence of  $k_n(p)$  follows from the following proposition.

**Proposition 5.1** *There exists an orthonormal system  $\{k_n\} \subset L_2(\mathbf{T}^\nu)$ , satisfying the conditions*

$$\begin{aligned} a) \quad &\text{supp } k_n \subset V_n(p_0), \\ b) \quad &\int_{V_n(p_0)} \int_{V_m(p_0)} \left( 1 + \int_{\mathbf{T}^\nu} \frac{v_3^2(t)dt}{(w_3(p, q, t) - z_0)^2} \right) \times \\ &\left[ \frac{v_{21}(p)(l_3 v_{22}(q) - l_1 v_2(q))}{\Delta_3^2(p, q; z_0)} + \frac{v_{21}(q)(l_3 v_{22}(p) - l_1 v_2(p))}{\Delta_3^2(p, q; z_0)} \right] k_n(p) k_m(q) dp dq = 0, \quad n \neq m. \end{aligned}$$

**Proof.** We construct the sequence  $\{k_n\}$  by induction. Let

$$k_1(p) = \chi_{V_1}(p) \left( \sqrt{\mu(V_1(p_0))} \right)^{-1}.$$

We choose the function  $\tilde{k}_2 \in L_2(V_2(p_0))$ , such that  $\|\tilde{k}_2\|_{\mathcal{H}_1} = 1$  and  $(\tilde{k}_2, \varepsilon_1^{(2)}) = 0$ , where

$$\begin{aligned} \varepsilon_1^{(2)}(p) &= \chi_{V_2}(p) \int_{\mathbf{T}^\nu} \left( 1 + \int_{\mathbf{T}^\nu} \frac{v_3^2(t)dt}{(w_3(p, q, t) - z_0)^2} \right) \times \\ &\left[ \frac{v_{21}(p)(l_3 v_{22}(q) - l_1 v_2(q))}{\Delta_3^2(p, q; z_0)} + \frac{v_{21}(q)(l_3 v_{22}(p) - l_1 v_2(p))}{\Delta_3^2(p, q; z_0)} \right] k_1(q) dq. \end{aligned}$$

We set  $k_2(p) = \tilde{k}_2(p)\chi_{V_2}(p)$  and continue the process. Assuming that the functions  $k_1(\cdot), \dots, k_n(\cdot)$  are constructed, we choose the function  $\tilde{k}_{n+1} \in L_2(V_{n+1}(p_0))$ , such that  $\|\tilde{k}_{n+1}\|_{\mathcal{H}_1} = 1$  and it is orthogonal to the functions

$$\varepsilon_i^{(n+1)}(p) = \chi_{V_{n+1}}(p) \int_{\mathbf{T}^\nu} \left( 1 + \int_{\mathbf{T}^\nu} \frac{v_3^2(t)dt}{(w_3(p, q, t) - z_0)^2} \right) \times$$

$$\left[ \frac{v_{21}(p)(l_3 v_{22}(q) - l_1 v_2(q))}{\Delta_3^2(p, q; z_0)} + \frac{v_{21}(q)(l_3 v_{22}(p) - l_1 v_2(p))}{\Delta_3^2(p, q; z_0)} \right] k_i(q) dq, \quad i = \overline{1, n}.$$

We set  $k_{n+1}(p) = \tilde{k}_{n+1}(p)\chi_{V_{n+1}}(p)$ . We have thus constructed an orthonormalized system  $\{k_n\}$  satisfying the conditions of the proposition. The Proposition 5.1 is proved.  $\square$

**We resume the proof of Theorem 2.1.**

We assume that  $\Delta_2(p_0; z_0) = 0$  and  $l_1 \neq 0, l_2 = 0$ . Then

$$\|f^{(n)}\|_{\mathcal{H}(0,3)}^2 \geq \|f_1^{(n)}\|_{\mathcal{H}_1}^2 = \frac{d_1}{\mu(V_n(p_0))}, \quad d_1 = l_1^2 > 0.$$

Let  $\Delta_2(p_0; z_0) = 0$  and  $l_3 \neq 0, l_1 = l_2 = 0$ . Then

$$\|f^{(n)}\|_{\mathcal{H}(0,3)}^2 \geq \|f_2^{(n)}\|_{\mathcal{H}_2}^2 = \frac{l_3^2}{\mu(V_n(p_0))} \int_{V_n(p_0)} \int_{\mathbf{T}^\nu} \left| \frac{v_{22}(q)k_n(p)}{\Delta_3(p, q; z_0)} \right|^2 dp dq \geq \frac{d_2}{\mu(V_n(p_0))},$$

$$d_2 = \frac{l_3^2 \|v_{22}\|_{\mathcal{H}_1}^2}{\max_{p, q \in \mathbf{T}^\nu} |\Delta_3(p, q; z_0)|^2}.$$

Similarly, we can prove that in the case where  $\Delta_1(p_0; z_0) = 0, l_2 \neq 0$  and  $l_1 = l_3 = 0$  the inequality

$$\|f^{(n)}\|_{\mathcal{H}(0,3)}^2 \geq \frac{d_3}{\mu(V_n(p_0))}, \quad d_3 = \frac{l_2^2 \|v_{21}\|_{\mathcal{H}_1}^2}{\max_{p, q \in \mathbf{T}^\nu} |\Delta_3(p, q; z_0)|^2}$$

holds. Therefore

$$\|f^{(n)}\|_{\mathcal{H}(0,3)}^2 \geq \frac{\xi_0}{\mu(V_n(p_0))}, \quad (5.2)$$

where

$$\xi_0 = \min\{d_1, d_2, d_3\} > 0.$$

We set  $\tilde{f}^{(n)} = f^{(n)} / \|f^{(n)}\|_{\mathcal{H}(0,3)}$ . It is clear that the system  $\{\tilde{f}^{(n)}\}$  is orthonormal.

We consider the operator  $(H - z_0)\tilde{f}^{(n)}$  and estimate its norm as

$$\|(H - z_0)\tilde{f}^{(n)}\|_{\mathcal{H}(0,3)} \leq \|A(z_0)\tilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}} + \|K(z_0)\tilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}},$$

where

$$\tilde{\psi}^{(n)} = \left( 0, \frac{\psi_1^{(n)}}{\|f^{(n)}\|_{\mathcal{H}(0,3)}}, \frac{\psi_2^{(n)}}{\|f^{(n)}\|_{\mathcal{H}(0,3)}}, \frac{\psi_3^{(n)}}{\|f^{(n)}\|_{\mathcal{H}(0,3)}} \right).$$

We note that  $\{\tilde{\psi}^{(n)}\} \subset \overline{\mathcal{H}}$  is bounded orthonormal system. Indeed, the orthogonality follows since for any  $n \neq m$  the supports of the functions  $\tilde{\psi}^{(n)}$  and  $\tilde{\psi}^{(m)}$  are nonintersecting. The equality

$$\|\tilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}}^2 = \frac{1}{\|f^{(n)}\|_{\mathcal{H}(0,3)}^2} \frac{1}{\mu(V_n(p_0))} (l_1^2 + l_2^2 + l_3^2)$$

and inequality (5.2) imply that the system  $\{\tilde{\psi}^{(n)}\}$  is uniformly bounded, i. e.,

$$\|\tilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}} \leq \frac{1}{\xi_0} \|\mathbf{1}\|_{\mathbf{C}^4}^2$$

for all positive integers  $n$ .

Since the operator  $K(z_0)$  is a compact, it follows that  $\|K(z_0)\tilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}} \rightarrow 0$  as  $n \rightarrow \infty$ .

We next estimate  $\|A(z_0)\tilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}}$ . Applying the Schwarz inequality we have

$$\|A(z_0)\tilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}}^2 \leq M^2 \sup_{p \in V_n(p_0)} \|A(p; z_0)\mathbf{1}\|_{\mathbf{C}^4}^2 \quad \text{with} \quad M^2 = \max\left\{\frac{2}{\xi_0}, \frac{\|v_{22}\|_{\mathcal{H}_1}^2}{\xi_0}, \frac{\|v_{21}\|_{\mathcal{H}_1}^2}{\xi_0}\right\}.$$

The continuity of the matrix-valued function  $A(\cdot; z_0)$  implies that

$$\sup_{p \in V_n(p_0)} \|A(p; z_0)\mathbf{1}\|_{\mathbf{C}^4} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

Therefore, for the sequence of orthonormal vector-functions  $\{\tilde{f}^{(n)}\}$  it follows that

$$\|(H - z_0)\tilde{f}^{(n)}\|_{\mathcal{H}^{(0,3)}} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

and hence by Weyl's criterion we have that  $z_0 \in \sigma_{ess}(H)$ . Since  $z_0$  is an arbitrary point of  $\sigma(H_1) \cup \sigma(H_2)$ , it follows that  $\sigma(H_1) \cup \sigma(H_2) \subset \sigma_{ess}(H)$ . Thus we have proved that  $\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3) \subset \sigma_{ess}(H)$ .

Now we prove the converse inclusion, that is,  $\sigma_{ess}(H) \subset \sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3)$ . Since for any  $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  the operator  $K(z)$  is a compact and  $A^{-1}(z)$  is bounded, we have that  $f(z) = A^{-1}(z)K(z)$  is a compact-valued analytic function in  $\mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ . From the self-adjointness of  $H$  and Lemma 4.2 it follows that the operator  $(\mathbf{I} - f(z))^{-1}$  exists for all  $\text{Im} z \neq 0$ , where  $\mathbf{I}$  is an identity operator in  $\overline{\mathcal{H}}$ . In accordance with the analytic Fredholm theorem, we conclude that the operator-valued function  $(\mathbf{I} - f(z))^{-1}$  exists on  $\mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$  everywhere except at a discrete set  $S$ , where it has finite-rank residues. Hence, with  $\sigma_{disc}(H)$  denoting the discrete spectrum of  $H$ , we have  $\sigma(H) \setminus \sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3) \subset \sigma_{disc}(H) = \sigma(H) \setminus \sigma_{ess}(H)$ , i. e.,  $\sigma_{ess}(H) \subset (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ . The Theorem 2.1 is completely proved.  $\square$

The proof of Theorem 2.2 follows from the Theorems 2.1 and 3.1.

## 6 BLOCK OPERATOR MATRICES AND A VARIATIONAL TECHNIQUE

In this section we give a variational technique to find the boundaries of  $\sigma_{ess}(H)$  and eigenvalues from some interior part of the discrete spectrum of  $\sigma(H)$ . Define

$$A = \begin{pmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{pmatrix}, \quad C = \begin{pmatrix} H_{22} & H_{23} \\ H_{32} & H_{33} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ H_{12} & 0 \end{pmatrix}, \quad B^* = \begin{pmatrix} 0 & H_{21} \\ 0 & 0 \end{pmatrix},$$

where  $A : \mathcal{H}^{(0,1)} \rightarrow \mathcal{H}^{(0,1)}$ ,  $C : \mathcal{H}^{(2,3)} \rightarrow \mathcal{H}^{(2,3)}$ ,  $B : \mathcal{H}^{(2,3)} \rightarrow \mathcal{H}^{(0,1)}$  and  $B^* : \mathcal{H}^{(0,1)} \rightarrow \mathcal{H}^{(2,3)}$ .

Then the operator  $H$  acting in Hilbert space  $\mathcal{H}^{(0,1)} \oplus \mathcal{H}^{(2,3)}$  defined by (2.1) can be written as a symmetric operator matrix in the form

$$H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}. \quad (6.1)$$

For any subspace  $S \subset \mathcal{H}^{(0,3)}$  define  $S^1 = \{x \in S : \|x\| = 1\}$  and let

$$\lambda_i(H) = \inf_{S \in S_i} \max_{S^1} (Hx, x)$$

and

$$\mu_i(H) = \sup_{S \in S_i} \min_{S^1} (Hx, x),$$

where  $S_i$  denotes the set of all subspaces of dimension  $i$ . These numbers, in general are not eigenvalues of  $H$ . Now define the boundaries of the spectrum and essential spectrum of  $H$  as

$$\begin{aligned} a(H) &= \min \sigma(H), \quad b(H) = \max \sigma(H), \\ a_{ess}(H) &= \min \sigma_{ess}(H), \quad b_{ess}(H) = \max \sigma_{ess}(H). \end{aligned}$$

Notice that, although we are mainly interested in spectral properties for the operator  $H$ , but the facts given below valid for a self-adjoint operator and symmetric operator matrices of the form (6.1).

It is known from the classical Courant-Hilbert-Weyl variational theory that, if  $a(H) < a_{ess}(H)$  then the spectrum of  $H$  in  $[a(H), a_{ess}(H))$  is discrete. Moreover, if eigenvalues in  $[a(H), a_{ess}(H))$  arranged in increasing order, including multiplicities, then they are equal to

$$\lambda_i(H) = \inf_{S \in S_i} \max_{S^1} (Hx, x), \quad i = 1, 2, \dots,$$

where the inf is attained, i. e., there exists a subspace  $S \in S_i$  such that

$$\lambda_i(H) = \max_{S^1} (Hx, x).$$

Now we give some results from the classical variational theory to obtain  $a_{ess}(H) = \min \sigma_{ess}(H)$ . First, if  $a_{ess}(H) = a(H)$  then it means that  $\min \sigma_{ess}(H) = \min \sigma(H)$  and for this reason we let  $a(H) < a_{ess}(H)$ . Two cases are possible (see [23], Theorem 1, p. 12):

I) The first case:  $\lambda_1(H), \lambda_2(H), \dots, \lambda_n(H)$  are attained but  $\lambda_{n+1}(H)$  is not attained, i. e.,

$$\lambda_i(H) = \min_{S \in S_i} \max_{S^1} (Hx, x),$$

for  $i = 1, 2, \dots, n$ ,

$$\lambda_{n+1}(H) = \inf_{S \in S_{n+1}} \max_{S^1} (Hx, x)$$

and no subspace  $S \in S_{n+1}$  such that

$$\lambda_{n+1}(H) = \max_{S^1} (Hx, x).$$

Then there are only  $n$  eigenvalues of  $H$  in  $[a(H), a_{ess}(H))$ , which can be described by

$$\lambda_i(H) = \min_{S \in S_i} \max_{S^1} (Hx, x), \quad i = 1, 2, \dots, n \quad (6.2)$$

and

$$\min \sigma_{ess}(H) := a_{ess}(H) = \lambda_{n+1} = \lambda_{n+2} = \dots \quad (6.3)$$

2) The second case: all of the numbers  $\lambda_i(H)$  are attained, i. e.,

$$\lambda_i(H) = \min_{S \in S_i} \max_{S^1} (Hx, x), \quad i = 1, 2, \dots, n, \dots$$

Then in this case the spectrum of  $H$  in  $[a(H), a_{ess}(H))$  consists of countable number of eigenvalues  $\lambda_i(H)$  and

$$a_{ess}(H) = \lim_{n \rightarrow \infty} \lambda_n(H). \quad (6.4)$$

The same results hold for  $b_{ess}(H) = \max \sigma_{ess}(H)$  but we need to replace  $\lambda_i(H)$  by  $\mu_i(H)$ .

Note that classical variational principles are applicable mainly to describe the discrete spectrum at the end parts of  $\sigma(H)$ . Dipper results can be obtained if we use the following operator function (operator pencil) technique. Here we give a method (see [4, 6, 7]) which allows to find eigenvalues in some interior parts of the discrete spectrum of symmetric block operator matrices (particularly, for  $H$ ) of the form (6.1).

First we give a short information on operator functions (see [4, 6, 7, 10]). Denote by  $S(\mathcal{H})$  the space of bounded symmetric operators on a Hilbert space  $\mathcal{H}$ . Let  $L(\lambda)$  be an operator valued function (or simply operator function), defined on an interval  $[\alpha, \beta]$  with values in  $S(\mathcal{H})$ . So,

$$L : [\alpha, \beta] \rightarrow S(\mathcal{H}).$$

A typical example is an operator polynomial of the form  $L(\lambda) = \lambda^n A_n + \lambda^{(n-1)} A_{n-1} + \dots + A_0$ , where  $A_i \in S(\mathcal{H})$ ,  $i = 0, 1, \dots, n$ . Polynomial operator functions are often called operator pencils. We denote the spectrum and the essential spectrum of  $L$  by  $\sigma(L)$  and  $\sigma_{ess}(L)$ , respectively. Define the family of functions  $\varphi_x(\lambda)$  depending on  $x \in \mathcal{H} \setminus \{0\}$  by

$$\varphi_x(\lambda) := (L(\lambda)x, x).$$

In the variational theory of operator functions it is always supposed the following two conditions are satisfied:

**I)** The equation  $\varphi_x(\lambda) = 0$  has at most one solution on  $[\alpha, \beta]$  (which is denoted by  $p(x)$ ) and  $\varphi_x(\lambda)$  is decreasing (or increasing) at  $p(x)$ , i. e.,

$$\varphi_x(\lambda) < 0 \Leftrightarrow \lambda > p(x),$$

$$\varphi_x(\lambda) > 0 \Leftrightarrow \lambda < p(x).$$

**II)**  $\kappa_\alpha(L) := \max \dim \{E \mid \varphi_x(\lambda) < 0, x \in E \setminus \{0\}\} < +\infty$ .

By these conditions if the equation  $\varphi_x(\lambda) = 0$  has no solution on  $[\alpha, \beta]$  for some  $x$  then either  $\varphi_x(\lambda) < 0$  or  $\varphi_x(\lambda) > 0$  for all  $\lambda \in [\alpha, \beta]$ . Clearly, the functional  $p(x)$  in general is not defined for all  $x \neq 0$  and in this case we define the extended functional  $p(x)$  as

$$p(x) = \begin{cases} \lambda_0, & \text{if } \varphi_x(\lambda_0) = 0, \\ +\infty, & \text{if } \varphi_x(\lambda) > 0, \lambda \in [\alpha, \beta], \\ -\infty, & \text{if } \varphi_x(\lambda) < 0, \lambda \in [\alpha, \beta]. \end{cases}$$

Now the same formulas (6.2), (6.3) and (6.4) (under the same conditions) hold (see [4], [10]) for the operator function  $L(\lambda)$  if we replace  $(Hx, x)$  by  $p(x)$  and define

$$\lambda_i(L) = \inf_{S \in S_i} \sup_{S^1} p(x)$$



and

$$\mu_i(L) = \sup_{S \in S_i} \inf_{S^1} p(x).$$

It means that in the spectral theory of operator functions the functional  $p(x)$ , which is called a Rayleigh functional, plays the same role as the quadratic form  $\frac{(Hx, x)}{(x, x)}$  of the operator  $H$  in the operator theory.

Let  $I^{(n, m)}$  be an identity operator in  $\mathcal{H}^{(n, m)}$ ,  $0 \leq n < m \leq 3$ .

Now we give a connection between the spectrum of  $H$  of the form (6.1) and the spectrum of the operator function defined below (see for details [4, 6, 7]). The first important step is

**Theorem 6.1** ([4, 6, 7]). *The spectrum of  $H$  of the form (6.1) outside of the spectrum  $C$  coincides with the spectrum of the operator function*

$$L(\lambda) = A - \lambda I^{(0,1)} - B(C - \lambda I^{(2,3)})^{-1} B^*.$$

Finally, we give a theorem which shows how one can obtain eigenvalues from some interior parts of the discrete spectrum of  $H$  by using the Rayleigh functional  $p(x)$  for  $L(\lambda)$  (see [4], pp. 204-205).

**Theorem 6.2** *Let  $H$  be an operator matrix of the form (6.1) and  $\sigma(C) < \sigma_{ess}(A)$ . Then,*

1) *the spectrum of  $H$  in  $(b(C), a_{ess}(A))$  is discrete with only possible accumulation point at  $a_{ess}(A)$ .*

2) *For  $L(\lambda) = A - \lambda I^{(0,1)} - B(C - \lambda I^{(2,3)})^{-1} B^*$  we have*

$$\sigma(L) \cap (b(C), a_{ess}(A)] = \sigma(H) \cap (b(C), a_{ess}(A)],$$

3) *if  $\lambda_i(H)$ ,  $i = 1, 2, 3, \dots$  are eigenvalues of  $H$  in  $(b(C), a_{ess}(A))$ , then*

$$\lambda_i(H) = \inf_{S \in S_i} \max_{S^1} p(x),$$

where  $p(x)$  is the Rayleigh functional for  $L(\lambda)$ .

**A sketch of the proof.** Evidently, it is enough to show that the operator function  $L(\lambda)$  satisfies the condition **I**) and **II**) on  $(b(C), a_{ess}(A))$ . It follows from the Hilbert identity  $R_\lambda(C) - R_\mu(C) = (\lambda - \mu)R_\lambda(C)R_\mu(C)$  that  $R'_\lambda(C) = R_\lambda^2(C)$ , where  $R_\lambda(C) := (C - \lambda I^{(2,3)})^{-1}$  is the resolvent of the operator  $C$ . Using this fact we have

$$L'(\lambda) = -I^{(1,2)} - B(C - \lambda I^{(2,3)})^{-2} B^* \ll 0$$

for all  $\lambda \in (b(C), a_{ess}(A)]$ . Here  $L'(\lambda) \ll 0$  means  $(L'(\lambda)x, x) \leq \delta(x, x)$  for all  $x$  and some  $\delta > 0$ .

In fact by the spectral theorem for a self-adjoint operator we can write

$$C = \int_{\sigma(C)} s dE_C(s),$$

where  $E_C(S)$  is the spectral measure of the operator  $C$ . Then

$$C - \lambda I^{(2,3)} = \int_{\sigma(C)} (s - \lambda) dE_C(s) < 0,$$

because  $s - \lambda < 0$  for  $\lambda > b(C)$ . Notice that the inequality  $C - \lambda I^{(2,3)} < 0$  also follows from the fact that the spectrum of a bounded operator is a subset of the closure of its numerical range. Now we have  $L'(\lambda) \ll 0$  and the condition **I**) follows from this inequality. On the other hand for  $\alpha = b(C)$  we can write

$$\{x \mid (L(\alpha)x, x) < 0\} \subset \{x \mid ((A - \alpha I^{(0,1)})x, x) < 0\}.$$

It follows from this that  $\kappa_\alpha(\lambda) \leq N(\alpha, A)$ , where  $N(\alpha, A)$  is the spectral distribution function of  $A$ . The condition  $\alpha = b(C) < \sigma_{ess}(A)$  means that  $N(\alpha, A)$  is finite and by the inequality  $\kappa_\alpha(\lambda) \leq N(\alpha, A)$  we get

$$\kappa_\alpha(\lambda) = \max\{E \mid L(\alpha)x, x) < 0, x \in E \setminus \{0\} < +\infty,$$

i. e., the condition **II**) is satisfied. Consequently, the eigenvalues of the operator  $H$  in the interval  $(b(C), a_{ess}(A))$  can be characterized by variational principles for the operator function  $L(\lambda)$  (see [4, 6, 7, 10]). More precisely,

$$\lambda_i(H) = \min_{S \in S_i} \sup_{S^1} p(x), \quad i = 1, 2, \dots,$$

where  $p(x)$  is the Rayleigh functional of the operator function  $L(\lambda) = A - \lambda I^{(0,1)} - B(C - \lambda I^{(2,3)})^{-1}B^*$  on  $[b(C), a_{ess}(A)]$ .

**ACKNOWLEDGEMENTS.** The first author would like to thank the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, for the kind hospitality and support and the Commission on Development and Exchanges of the International Mathematical Union for travel grant.

## References

- [1] S. Albeverio, S. N. Lakaev and J. I. Abdullaev. On the Finiteness of the Discrete Spectrum of a Four-Particle Lattice Schrödinger Operator. *Func. Anal. Appl.* **36** (2002), No. 3, 212–216.
- [2] Sergio Albeverio, Saidakhmat N. Lakaev and Tulkin H. Rasulov. On the Spectrum of an Hamiltonian in Fock Space. *Discrete Spectrum Asymptotics. J. Stat. Phys.* **127** (2007), No. 2, 191–220.
- [3] Sergio Albeverio, Saidakhmat N. Lakaev and Tulkin H. Rasulov. The Efimov Effect for a Model Operator Associated with the Hamiltonian of a non Conserved Number of Particles. *Methods Func. Anal. Topol.* **13** (2007), No. 1, 1–16.
- [4] P. Binding, D. Eschwe, H. Langer. Variational principles for real eigenvalues of self-adjoint operator pencils. *Integr. Equ. Oper. Theory.* **38** (2000), No. 2, 190–206.
- [5] H. L. Cycon, R. Froese, W. Kirsch and B. Simon. *Schrödinger operators with applications to quantum mechanics and global geometry.* Springer, Berlin-Heidelberg-New York, 1987.
- [6] D. Eschwe, H. Langer. Triple variational principles for eigenvalues of self-adjoint operators and operator functions. *SIAM J. Math. Anal.* **34** (2002), No. 1, 228–238.

- [7] D. Eschwe, H. Langer. Variational principles for eigenvalues of self-adjoint operator functions. *Integr. Equ. Oper. Theory*. **49** (2004), 287–321.
- [8] L. D. Faddeev and S. P. Merkuriev. Quantum scattering theory for several particle systems. Kluwer Academic Publishers, 1993.
- [9] K. O. Friedrichs. Perturbation of spectra in Hilbert space. Amer. Math. Soc. Providence, Rhode Island, 1965.
- [10] M. Hasanov. An approximation method in the variational theory of the spectrum of operator pencils. *Acta Appl. Math.* **71** (2002), No. 2, 117–126.
- [11] W. Hunziker. On the spectra of Schrödinger multiparticle Hamiltonians. *Helv. Phys. Acta*. **39** (1966), 451–462.
- [12] S. N. Lakaev and T. Kh. Rasulov. A Model in the Theory of Perturbations of the Essential Spectrum of Multiparticle Operators. *Math. Notes*. **73** (2003), No. 4, 521–528.
- [13] S. N. Lakaev and T. Kh. Rasulov. Efimov’s Effect in a Model of Perturbation Theory of the Essential Spectrum. *Func. Anal. Appl.* **37** (2003), No. 1. 69–71.
- [14] R. Minlos and H. Spohn. The Three-Body Problem in Radioactive Decay: The Case of One Atom and At Most Two Photons. *Amer. Math. Soc. Transl.* **177** (1996), No. 2, 159–193.
- [15] A. I. Mogilner. Hamiltonians in solid state physics as multiparticle discrete Schrödinger operators: problems and results. *Advances in Sov. Math.* **5** (1991), 139–194.
- [16] M. É. Muminov. A Hunziker-van Winter-Zhislin theorem for a four-particle lattice Schrödinger operator. *Theor. Math. Phys.* **148** (2006), No. 3, 1236–1250.
- [17] V. S. Rabinovich. On the Essential Spectrum of Electromagnetic Schrödinger operators. *Israel Mathematics Conference Proceedings ”Complex Analysis and Dynamical Systems II”*. Nahariya, Israel, June 9-12, 2003, 331–343.
- [18] Vladimir S. Rabinovich and Steffen Roch. The essential spectrum of Schrödinger operators on lattices. *J. Phys. A: Math. Gen.* **39** (2006), 8377–8394.
- [19] T. Kh. Rasulov. Discrete Spectrum of a Model Operator in Fock Space. *Theor. Math. Phys.* **152** (2007), No. 3, 1313–1321.
- [20] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. IV: Analysis of Operators*. Academic Press, New York, 1979.
- [21] I. M. Sigal, A. Soffer and L. Zielinski. On the spectral properties of Hamiltonians without conservation of the particle number. *J. Math. Phys.* **42** (2002), No. 4, 1844–1855.
- [22] G. R. Yodgorov and M. É. Muminov. Spectrum of a Model Operator in the Perturbation Theory of the Essential Spectrum. *Theor. Math. Phys.* **144** (2005), No. 3, 1344–1352.

- [23] A. Weinstein, W. F. Stenger. Methods of Intermediate Problems for Eigenvalues. Academic Press, 1972.
- [24] C. Van Winter. Theory of finite systems of particles. I. Mat. Fys. Skr. Danske Vid. Selsk. **1** (1964), 1–60.
- [25] G. M. Zhislin. Investigations of the spectrum of the Schrödinger operator for a many body system. Trudy Moskov. Mat. Obshch. **9** (1960), 81–128.
- [26] Yu. Zhukov and R. Minlos. Spectrum and scattering in a "spin-boson" model with not more than three photons. Theor. Math. Phys. **103** (1995), No. 1, 398–411.

TULKIN H. RASULOV  
 SAMARKAND STATE UNIVERSITY, DEPARTMENT OF PHYSICS AND MATHEMATICS, 15  
 UNIVERSITY BOULEVARD, SAMARKAND, 140104, UZBEKISTAN  
 E-mail: tulkin\_rasulov@yahoo.com

MUKHIDDIN I. MUMINOV  
 SAMARKAND STATE UNIVERSITY, DEPARTMENT OF PHYSICS AND MATHEMATICS, 15  
 UNIVERSITY BOULEVARD, SAMARKAND, 140104, UZBEKISTAN  
 E-mail: mmuminov@mail.ru

MAHIR HASANOV  
 SOUTHERN ALBERTA INSTITUTE OF TECHNOLOGY, 1301 16TH AVE NW CALGARY,  
 ALBERTA, CANADA T2M 0L4  
 E-mail: hasanov61@yahoo.com